

Small entries of neutrino mass matrices

E. Kh. Akhmedov *

*Centro de Física das Interações Fundamentais (CFIF)
Departamento de Física, Instituto Superior Técnico
Av. Rovisco Pais, P-1049-001 Lisboa, Portugal*

Abstract

We consider phenomenologically allowed structures of the neutrino mass matrix in the case of three light neutrino species. Constraints from the solar, atmospheric and reactor neutrino experiments as well as those from the neutrinoless double beta decay are taken into account. Both hierarchical and quasi-degenerate neutrino mass cases are studied. Assuming maximal $\nu_\mu - \nu_\tau$ mixing we derive simple approximate expressions giving the values of the neutrino masses and remaining lepton mixing angles in terms of the entries of the neutrino mass matrix. Special attention is paid to the small entries which are usually not specified in discussions of the neutrino mass matrix textures. We specifically discuss the stability of neutrino masses and lepton mixing angles with respect to the choice and variations of these small entries.

*On leave from National Research Centre Kurchatov Institute, Moscow 123182, Russia. E-mail: akhmedov@gtae2.ist.utl.pt

1 Introduction

Recent data from the Super-Kamiokande experiment [1, 2, 3] give a strong support to the interpretation of the atmospheric [4] and solar [5] neutrino problems in terms of neutrino oscillations. The results of the solar and atmospheric neutrino experiments along with the constraints from the reactor and accelerator experiments provided us with an important information on neutrino masses and mixing. This has resulted in an increased theoretical activity in the field with a large number of publications dedicated to the reconstruction of the neutrino masses and mixing and building theoretical models capable of producing the requisite neutrino mass matrices (see e.g. [6, 7, 8, 9, 10]).

In most of these studies some special structures or textures are suggested for the neutrino mass matrix. The special structures contain a number of large entries and also some small entries which are replaced by zeros in the textures. It is usually assumed that this form of the neutrino mass matrices emerge as a result of a flavour symmetry. In the exact symmetry limit some entries of the neutrino mass matrix vanish, but they are supposed to be filled in with small nonzero terms after a small breaking of the flavour symmetry is allowed. In many studies, however, these small entries of the neutrino mass matrices are not specified or are given only by an order of magnitude.

While the neutrino mass textures reproduce correctly the gross features of the neutrino mass matrix, the small terms are of crucial importance for obtaining a realistic pattern of neutrino masses and mixing. This includes not only reproducing the correct values of the neutrino mass squared differences and mixing angles but also the correct correspondence between the ordering of the neutrino masses and the values of the mixing angles. Therefore any realistic neutrino mass model has to address the issue of the small entries of the neutrino mass matrices.

In the present paper we consider the phenomenologically allowed forms of the neutrino mass matrices in the case of three light neutrino species, paying special attention to the small entries. The analysis is performed without resorting to any specific model of neutrino mass and therefore is quite general. Both hierarchical and quasi-degenerate neutrino mass cases are considered. Assuming maximal ν_μ - ν_τ mixing, as suggested by the Super-Kamiokande atmospheric neutrino data, we derive simple approximate formulas which give the values of neutrino masses and remaining mixing angles in terms of the entries of the neutrino mass matrix. They allow one to find the neutrino masses and mixings off hand, without performing an explicit diagonalization of the neutrino mass matrix in each particular case. We discuss the stability of neutrino masses and mixing with respect to the choice and variations of the small entries of the neutrino mass matrices. The question of stability is an important issue in discussions of the radiative corrections in various neutrino mass models [9]. In the present paper it is studied purely phenomenologically, and the results can be used in the analyses of any models with three light neutrinos species.

2 Experimental input

Currently there are three kinds of experimental indications of nonzero neutrino mass. These come from the solar neutrino experiments [5, 2], atmospheric neutrino experiments [4, 1] and the accelerator LSND data [11]. The LSND result is the only one that has not yet been independently confirmed and we therefore choose not to consider it here. The results of the solar and atmospheric neutrino experiments can then be described in terms of only three light neutrino species, ν_e , ν_μ and ν_τ , without introducing sterile neutrinos.

The solar neutrino deficit can be explained through the $\nu_e \rightarrow \nu_x$ oscillations where ν_x is ν_μ , ν_τ , or their linear combination. There are three main types of solutions of the solar neutrino problem: small mixing angle MSW effect (SMA), large mixing angle MSW effect (LMA) and vacuum neutrino oscillations (VO). The atmospheric neutrino data can be explained through the $\nu_\mu \rightarrow \nu_\tau$ oscillations with possible $\nu_\mu \leftrightarrow \nu_e$ oscillations as a subdominant channel. The corresponding allowed values of the neutrino parameters can be summarized as follows [12, 3]:

$$\begin{aligned}
\text{SMA :} & \quad \Delta m_\odot^2 \simeq (4 - 10) \cdot 10^{-6} \text{ eV}^2, & \sin^2 2\theta_\odot \simeq (0.1 - 1.0) \cdot 10^{-2} \\
\text{LMA :} & \quad \Delta m_\odot^2 \simeq (2 - 20) \cdot 10^{-5} \text{ eV}^2, & \sin^2 2\theta_\odot \simeq 0.65 - 0.97 \\
\text{VO :} & \quad \Delta m_\odot^2 \simeq (0.5 - 5) \cdot 10^{-10} \text{ eV}^2, & \sin^2 2\theta_\odot \simeq 0.6 - 1.0 \\
\text{Atm :} & \quad \Delta m_{atm}^2 \simeq (2 - 6) \cdot 10^{-3} \text{ eV}^2, & \sin^2 2\theta_{atm} \simeq 0.82 - 1.0
\end{aligned} \tag{1}$$

We shall use the parametrization of the unitary 3×3 matrix U describing the lepton mixing which coincides with the standard quark mixing matrix parametrization [13]. One can then identify the lepton mixing angles in (1) as $\theta_\odot = \theta_{12}$, $\theta_{atm} = \theta_{23}$.

In addition to the results listed above, there is an important limit on the element U_{e3} of the lepton mixing matrix coming from the CHOOZ reactor neutrino experiment [14], $|U_{e3}|^2(1 - |U_{e3}|^2) < 0.045 - 0.02$ for $\Delta m_{31}^2 \equiv \Delta m_{atm}^2$ ranging in the Super-Kamiokande allowed region $(2 - 6) \cdot 10^{-3} \text{ eV}^2$, which together with the solar neutrino observations means

$$\sin^2 \theta_{13} \equiv |U_{e3}|^2 \leq (0.047 - 0.02) \quad \text{for} \quad \Delta m_{31}^2 = (2 - 6) \cdot 10^{-3} \text{ eV}^2. \tag{2}$$

In the case of quasi-degenerate neutrinos we shall be also using the limit on the effective Majorana mass term m_{eff} of ν_e which comes from the Heidelberg-Moscow double beta decay experiment [15]:

$$m_{eff} \lesssim 0.2 \text{ eV}. \tag{3}$$

This effective mass is essentially the m_{ee} entry of the neutrino mass matrix.

The results of the existing neutrino experiments can therefore be summarized as follows. There are two distinct mass squared difference scales, Δm_\odot^2 and Δm_{atm}^2 , with the hierarchy $\Delta m_\odot^2 \ll \Delta m_{atm}^2 \sim 10^{-3} \text{ eV}^2$. The mixing angle θ_{23} responsible for the dominant channel of the atmospheric neutrino oscillations is close to the maximal one (45°), whereas the mixing angle θ_{12} which governs the solar neutrino oscillations can be either small or large; for the

VO solution it can also be very close or equal to the maximal one. The mixing angle θ_{13} responsible for the long-baseline $\nu_e \leftrightarrow \nu_x$ oscillations and the subdominant $\nu_\mu \leftrightarrow \nu_e$ oscillations of atmospheric neutrinos is either small or zero.

3 Neutrino mass matrices

The experimental information on the neutrino masses and lepton mixing angles allows one to reconstruct the phenomenologically allowed form of the neutrino mass matrix. We shall adopt the following convention. The neutrino mass eigenstate separated from the other two by the large Δm_{atm}^2 is ν_3 whereas those responsible for the solar neutrino problem which are separated by the small Δm_\odot^2 are ν_1 and ν_2 . In other words, $\Delta m_{atm}^2 \simeq \Delta m_{31}^2 \simeq \Delta m_{32}^2$, $\Delta m_\odot^2 = \Delta m_{21}^2$. Neutrino mass spectrum can be either hierarchical ($m_1, m_2 \ll m_3$ or $m_1, m_2 \gg m_3$) or quasi-degenerate, $m_1 \approx m_2 \approx m_3$, with only mass squared differences being hierarchical. The ordering of the ν_1 and ν_2 states is unimportant in the case of VO solution of the solar neutrino problem but it does make a difference in the case of the SMA and LMA solutions: the lower-mass state must have a larger ν_e component. This condition will prove to be very important in deriving the constraints on the entries of the neutrino mass matrices. We shall disregard possible CP violation effects in the leptonic sector and assume the neutrino mass matrix to be real. Its eigenvalues, which we hereafter denote m_i ($i = 1, 2, 3$), can be of either sign, depending of the relative CP parities of neutrinos. The physical neutrino masses are $|m_i|$.

We shall be assuming that the mixing angle responsible for the atmospheric neutrino oscillations $\theta_{23} = 45^\circ$ which is the best fit value of the Super-Kamiokande data [3]. The case when θ_{23} is close 45° but not exactly equal to this value can be treated similarly. In the first order in the small $\sin \theta_{13} \equiv \epsilon$ the lepton mixing matrix takes the form

$$U = \begin{pmatrix} c & s & \epsilon \\ -\frac{1}{\sqrt{2}}(s + c\epsilon) & \frac{1}{\sqrt{2}}(c - s\epsilon) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(s - c\epsilon) & -\frac{1}{\sqrt{2}}(c + s\epsilon) & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (4)$$

where $c \equiv c_{12}$, $s \equiv s_{12}$, and the neutrino flavour basis is $(\nu_e, \nu_\mu, \nu_\tau)$. The neutrino mass matrix in the basis where the charged lepton mass matrix has been diagonalized can be written as $m_L = U \text{diag}(m_1, m_2, m_3) U^T$ which gives

$$m_L = \begin{pmatrix} \mu & \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) + m_-cs] & \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) - m_-cs] \\ \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) + m_-cs] & \frac{1}{2}(m_3 + \mu' - 2m_-cs\epsilon) & \frac{1}{2}(m_3 - \mu') \\ \frac{1}{\sqrt{2}}[\epsilon(m_3 - \mu) - m_-cs] & \frac{1}{2}(m_3 - \mu') & \frac{1}{2}(m_3 + \mu' + 2m_-cs\epsilon) \end{pmatrix}. \quad (5)$$

Here

$$\mu = m_1c^2 + m_2s^2, \quad \mu' = m_1s^2 + m_2c^2, \quad m_- = m_2 - m_1, \quad (6)$$

We shall be using eq. (5) to derive the phenomenologically allowed forms of the neutrino mass matrix for various neutrino mass hierarches.

By making use of eqs. (5) and (6), which are valid up to the small corrections of the order $\epsilon^2 \lesssim 0.03$, one can readily derive the neutrino mass eigenvalues and lepton mixing angles in terms of the entries m_{ij} of the neutrino mass matrix m_L :

$$m_{1,2} = \frac{1}{2} \left\{ (m_{22} + m_{33})/2 + m_{11} - m_{23} \pm \sqrt{[(m_{22} + m_{33})/2 - m_{11} - m_{23}]^2 + 2(m_{12} - m_{13})^2} \right\}, \quad (7)$$

$$m_3 = \frac{1}{2}(m_{22} + m_{33} + 2m_{23}), \quad (8)$$

$$\tan 2\theta_{12} = \frac{\sqrt{2}(m_{12} - m_{13})}{\frac{1}{2}(m_{22} + m_{33}) - (m_{11} + m_{23})}, \quad \epsilon = \frac{\sqrt{2}(m_{12} + m_{13})}{m_{22} + m_{33} + 2m_{23} - 2m_{11}}. \quad (9)$$

We shall now consider various cases of interest.

3.1 Hierarchy $|m_1|, |m_2| \ll |m_3|$

In this case the neutrino mass matrix (5) takes the form ¹

$$m_L = m_0 \begin{pmatrix} \kappa & \varepsilon & \varepsilon' \\ \varepsilon & 1 + \delta - \delta' & 1 - \delta \\ \varepsilon' & 1 - \delta & 1 + \delta + \delta' \end{pmatrix} \quad (10)$$

with small $\kappa, \varepsilon, \varepsilon', \delta$ and δ' . One can now use eqs. (7)-(9) to find neutrino masses and mixing angles up to the corrections of the order ϵ^2 . It is not difficult, however, to find more accurate expressions, which include terms $\mathcal{O}(\epsilon^2)$, directly from the matrix (10) ²:

$$m_{1,2} \simeq \left(\frac{\kappa}{2} - \frac{\varepsilon^2 + \varepsilon'^2}{4} \right) + \left(\delta - \frac{\delta'^2}{4} \right) \pm \sqrt{\left[\left(\frac{\kappa}{2} - \frac{\varepsilon^2 + \varepsilon'^2}{4} \right) - \left(\delta - \frac{\delta'^2}{4} \right) \right]^2 + \frac{1}{2} \left[(\varepsilon - \varepsilon') + \frac{\delta'}{2}(\varepsilon + \varepsilon') \right]^2}, \quad (11)$$

$$m_3 \simeq 2 + \frac{\varepsilon^2 + \varepsilon'^2 + \delta'^2}{2} \simeq 2, \quad (12)$$

$$\tan 2\theta_{12} \simeq \frac{(\varepsilon - \varepsilon') + \frac{\delta'}{2}(\varepsilon + \varepsilon')}{\sqrt{2} \left[\left(\delta - \frac{\delta'^2}{4} \right) - \left(\frac{\kappa}{2} - \frac{\varepsilon^2 + \varepsilon'^2}{4} \right) \right]}, \quad \sin \theta_{13} \equiv \epsilon \simeq \frac{\varepsilon + \varepsilon'}{2\sqrt{2}(1 - \kappa/2)}. \quad (13)$$

¹Up to possible trivial sign changes due to the rephasing of the neutrino fields.

²While (5) is valid only up to and including the terms $\mathcal{O}(\epsilon)$, the form of the mass matrix (10) is generic for the case $|m_1|, |m_2| \ll |m_3|, |\epsilon| \ll 1$.

where the eigenvalues m_i are given in units of m_0 . The value of m_0 can be fixed through $\Delta m_{31}^2 \simeq (2m_0)^2 \simeq \Delta m_{atm}^2$. The mixing angle θ_{13} is small by construction and the limit (2) can be easily satisfied. The parameters $\Delta m_{21}^2 = \Delta m_{\odot}^2$ and θ_{12} are fully determined by the small entries of the neutrino mass matrix, the values of which should be chosen in accordance with the assumed solution of the solar neutrino problem.

For the MSW solutions (both SMA and LMA), the lower-mass state out of the two eigenstates, ν_1 and ν_2 , must have the larger ν_e component. The condition for this is

$$|4\delta - \delta'^2| > |2\kappa - (\varepsilon^2 + \varepsilon'^2)|. \quad (14)$$

Therefore the points in the parameter space at which the inequality in eq. (14) is replaced by the equality are unstable: a small change of the values of the small parameters can affect drastically the pattern of neutrino masses and mixing. One of these points, $4\delta - \delta'^2 = 2\kappa - (\varepsilon^2 + \varepsilon'^2)$, is the point at which the denominator in the expression for $\tan 2\theta_{12}$ in eq. (13) develops a pole, i.e. the mixing becomes maximal. Although at the other point, $4\delta - \delta'^2 = -[2\kappa - (\varepsilon^2 + \varepsilon'^2)]$, the value of $\tan 2\theta_{12}$ remains finite, it is an instability point as well. This can be explained as follows. At this point one has $m_2 = -m_1$ so that the eigenstates ν_1 and ν_2 are degenerate. Crossing this point in the parameter space would interchange the relative position of the two eigenstates – the lower mass eigenstate would become the higher mass one and vice versa. At the same time, their flavour composition remains nearly constant since both θ_{13} and θ_{12} are regular at this instability point. This means that the condition that the lower-mass state out of the two eigenstates have the larger ν_e component is violated on one of the sides of this instability point.

It should be noted that the expression $4\delta - \delta'^2$ on left hand side of eq. (14) is the determinant of the 2×2 matrix in the 23 subspace of the matrix (10). Therefore in models in which this subdeterminant exactly vanishes the condition (14) is not satisfied and the MSW solutions of the solar neutrino problem are not possible.

3.2 Inverted hierarchy, $|m_3| \ll |m_1| \approx |m_2|$

In the case $|m_3| \ll |m_1|, |m_2|$ the neutrino mass matrix (5) takes the form

$$m_L = m_0 \begin{pmatrix} \kappa & \varepsilon & \varepsilon' \\ \varepsilon & 1 + \delta - \delta' & -1 + \delta \\ \varepsilon' & -1 + \delta & 1 + \delta + \delta' \end{pmatrix}, \quad (15)$$

and the requirement $|m_1| \approx |m_2|$ leads to some additional constraints. There are essentially three possibilities:

(1) $\varepsilon, \varepsilon', \delta$ and δ' small, $\kappa \approx 2$. This gives $m_1 \simeq m_2$. The neutrino mass eigenvalues and mixing angles are given by

$$m_{1,2} \simeq m_0 \left[1 + \frac{\kappa}{2} \pm \sqrt{\left(1 - \frac{\kappa}{2}\right)^2 + \frac{(\varepsilon - \varepsilon')^2}{2}} \right], \quad m_3 \simeq \left(2\delta - \frac{\delta'^2}{2}\right) m_0, \quad (16)$$

$$\tan 2\theta_{12} \simeq \frac{(\varepsilon - \varepsilon')}{\sqrt{2}(1 - \kappa/2)}, \quad \sin \theta_{13} = \epsilon \simeq \frac{\varepsilon + \varepsilon'}{2\sqrt{2}(\delta - \kappa/2)}. \quad (17)$$

(2) ε , ε' , δ and δ' small, $\kappa \approx -2$. This gives $m_2 \simeq -m_1$. Formulas (16) and (17) apply in this case, too.

The case $m_2 \simeq -m_1$ obtains also when ε and ε' are not small provided they are nearly negative of each other. In this case ($\varepsilon' \approx -\varepsilon$, $|\varepsilon| \lesssim 1$, δ and δ' small, $\kappa \approx -2$) the values of $m_{1,2}$ and $\tan 2\theta_{12}$ are again given by the first equations in (16) and (17), whereas for m_3 and $\sin \theta_{13}$ one obtains

$$m_3 \simeq \frac{(\varepsilon + \varepsilon')^2 + (\varepsilon - \varepsilon')^2 \delta + (\varepsilon^2 - \varepsilon'^2) \delta' - 4\kappa \delta}{(\varepsilon^2 + \varepsilon'^2) - 2\kappa(1 + \delta) - 4\delta} m_0, \quad (18)$$

$$\sin \theta_{13} \simeq - \frac{(\varepsilon + \varepsilon') - \delta(\varepsilon - \varepsilon') - \varepsilon'(\delta' + m_3/m_0)}{\sqrt{\varepsilon^2(\varepsilon^2 + \varepsilon'^2) - 2\varepsilon(\varepsilon - \varepsilon')(\kappa - m_3/m_0) + 2(\kappa - m_3/m_0)^2}}. \quad (19)$$

In both cases (1) and (2) the value of m_0 can be fixed through $m_1^2 \simeq m_2^2 \simeq (2m_0)^2 \simeq \Delta m_{atm}^2$, whereas $\Delta m_{21}^2 = \Delta m_{\odot}^2 \simeq 8m_0^2 r$ where r is the square root that appears in the first equation in (16). The mixing angle θ_{13} is small by construction; in case (1) the value of $\tan 2\theta_{12}$ depends on how close κ is to 2 as compared to $\varepsilon - \varepsilon'$; in case (2) $\tan 2\theta_{12}$ is small when ε and ε' are small but can be large in the case $\varepsilon' \simeq -\varepsilon$, $|\varepsilon| \gtrsim 1$.

The condition that the lower-mass state out of the eigenstates ν_1 and ν_2 have the larger ν_e component, which is important for the SMA and LMA solutions of the solar neutrino problem, is

$$|\kappa| < 2. \quad (20)$$

It constrains only a large entry of the mass matrix and not the small entries. This is related to the particular choice of the parametrization of the mass matrix in eq. (15) which is convenient because it leads to the very simple expressions (16) and (17). In general, this condition reads $2|m_{11}| < |m_{22} + m_{33} - 2m_{23}|$, which includes both small and large entries. The parameter κ is required to be close to ± 2 in the cases under discussion. A small change of its value can drastically alter the pattern of the neutrino masses and mixing. As an example, consider the case $\kappa = 1.998$, $\varepsilon = 5 \cdot 10^{-3}$, $\varepsilon' = 1 \cdot 10^{-3}$, $\delta = 1 \cdot 10^{-2}$ and $\delta' = 3 \cdot 10^{-3}$. This yields $m_1 \simeq 1.996$, $m_2 \simeq 2.002$, $m_3 \simeq 0.02$ (in units of m_0), $\sin \theta_{13} \simeq -2.14 \cdot 10^{-3}$, $\cos \theta_{12} \simeq 0.816$ ($\sin^2 2\theta_{12} \simeq 0.889$). This choice of parameters is suitable for the LMA solution of the solar neutrino problem. The fact that $\cos^2 \theta_{12} > 1/2$ means that the lower-mass state ν_1 has the larger ν_e component than the higher mass state ν_2 , as it should. If, however, the value of κ is increased by just 0.2% while the other parameters are kept intact, the eigenstates ν_1 and ν_2 interchange their position so that now $m_1 > m_2$ (alternatively, one can rename the states in such a way that m_1 is always smaller than m_2 but then $\sin \theta_{12}$ and $\cos \theta_{12}$ interchange). This means that the lower-lying of the two mass eigenstates no longer has the larger ν_e component and therefore there is no solution to the solar neutrino problem at all. Similar situation takes place in case (2) when $\kappa \simeq -2$.

We have seen that, like in the case of the normal hierarchy $|m_{1,2}| \ll |m_3|$, in cases (1) and (2) of the inverted hierarchy there are instability regions which are close to the certain points in the parameter space. In the case of the normal hierarchy these instability regions are just small domains of the full allowed parameter space, and the parameters can in general be quite far from these domains. On the contrary, in the cases of the inverted hierarchy under discussion, the parameter κ *must* be close to ± 2 , so that the situation is generically unstable. This is related to the fact that the states ν_1 and ν_2 are quasi-degenerate in this case while in the case of the normal hierarchy their masses may be hierarchical.

(3) $\varepsilon' \simeq \pm \varepsilon$, $|\varepsilon| \gg 1$, $|\delta|, |\delta'|; |\kappa| \lesssim 1$. δ and δ' need not be small. This case also leads to $m_2 \simeq -m_1$. By a rescaling of the parameters it can also be formulated as $\varepsilon' \simeq \pm \varepsilon = \mathcal{O}(1)$, the rest of the entries of the matrix m_L being small. This case was suggested in [7] on the basis of the approximate $L_e - L_\mu - L_\tau$ symmetry. It is more convenient to use a different parametrization of the mass matrix in this case:

$$m_L = m_0 \begin{pmatrix} \varepsilon & 1 & \pm 1 \\ 1 & \delta & \varepsilon' \\ \pm 1 & \varepsilon' & \delta' \end{pmatrix}, \quad (21)$$

where ε , ε' , δ and δ' are small. The neutrino masses and mixing angles are now

$$m_1 \simeq m_0 \left\{ \frac{1}{4} [2(\varepsilon \pm \varepsilon') + \delta + \delta'] - \sqrt{2} \right\}, \quad m_2 \simeq m_0 \left\{ \frac{1}{4} [2(\varepsilon \pm \varepsilon') + \delta + \delta'] + \sqrt{2} \right\}, \quad (22)$$

$$m_3 \simeq \frac{m_0}{2} (\delta + \delta' \mp 2\varepsilon'), \quad \sin \theta_{13} \simeq \mp \frac{\delta - \delta'}{2\sqrt{2}}. \quad (23)$$

where the upper and lower signs refer to the corresponding signs of the matrix element $(m_L)_{13}$ in (21). In this case, too, the mixing angle θ_{13} is automatically small. The mixing angle $\theta_{12} = 45^\circ$ up to corrections of the second order in the small entries of the mass matrix, i.e. only the VO solution of the solar neutrino problem is possible³. This is an example of the so-called bimaximal mixing [8]. Unlike cases (1) and (2) discussed above, this case is very stable with respect to variations of the entries of the neutrino mass matrix. This is because the maximal mixing $\theta_{12} = 45^\circ$ makes the ordering of ν_1 and ν_2 unimportant.

3.3 Quasi-degenerate neutrinos, $|m_1| \simeq |m_2| \simeq |m_3|$

The quasi-degenerate cases can be classified according to the relative signs of the neutrino mass eigenvalues.

(1) $m_1 \simeq m_2 \simeq m_3$. We shall define the mass splittings $\tilde{\delta}$ and Δ through $m_2 = m_1 + \tilde{\delta}$, $m_3 = m_1 + \Delta$, $|\tilde{\delta}| \ll |\Delta| \ll |m_1|$. Then the neutrino mass matrix (5) can be written as

$$m_L = m_1 \text{diag}(1, 1, 1) + \frac{\Delta}{2} M, \quad (24)$$

³This remains true also in a more general case in which $(m_L)_{13} = A$, $(m_L)_{13} = B$ and for $A \sim B$ the mixing angle $\theta_{23} \simeq -\arctan(B/A)$ is large but not necessarily maximal.

where the matrix M has the same form as the one on the right hand side of eq. (10). Therefore the formulas of sec. 3.1 apply to this case up to the obvious modification of the mass eigenvalues. Zeroth order texture in this case is proportional to the unit matrix.

(2) $m_1 \simeq m_2 \simeq -m_3$. We now denote $m_2 = m_1 + \tilde{\delta}$, $m_3 = -m_1 + \Delta$. The mass matrix (5) takes the form

$$m_L = m_1 \begin{pmatrix} 1 & -\sqrt{2}\epsilon & -\sqrt{2}\epsilon \\ -\sqrt{2}\epsilon & 0 & -1 \\ -\sqrt{2}\epsilon & -1 & 0 \end{pmatrix} + \frac{\Delta}{2} \begin{pmatrix} 0 & \sqrt{2}\epsilon & \sqrt{2}\epsilon \\ \sqrt{2}\epsilon & 1 & 1 \\ \sqrt{2}\epsilon & 1 & 1 \end{pmatrix} + \frac{\tilde{\delta}}{2} \begin{pmatrix} 2s^2 & \sqrt{2}(cs - \epsilon s^2) & -\sqrt{2}(cs + \epsilon s^2) \\ \sqrt{2}(cs - \epsilon s^2) & c^2 - 2cs\epsilon & -c^2 \\ -\sqrt{2}(cs + \epsilon s^2) & -c^2 & c^2 + 2cs\epsilon \end{pmatrix}, \quad (25)$$

where the notation is the same as in eq. (5). The zeroth order texture is given by the first term in this equation in which ϵ is set equal to zero. Neutrino masses and mixing angles can be expressed through the entries of the mass matrix (25) with the use of eqs. (7)-(9).

(3) $m_1 \simeq -m_2 \simeq -m_3$ ⁴. We now denote $m_2 = -m_1 + \tilde{\delta}$, $m_3 = -m_1 + \Delta$. The mass matrix (5) takes the form

$$m_L = M_1 + M_2 + M_3, \quad (26)$$

where M_2 and M_3 coincide with the second and the third terms on the right hand side of eq. (25) respectively, and

$$M_1 = -m_1 \begin{pmatrix} -(c^2 - s^2) & \sqrt{2}(cs + \epsilon c^2) & -\sqrt{2}(cs - \epsilon c^2) \\ \sqrt{2}(cs + \epsilon c^2) & c^2 - 2cs\epsilon & s^2 \\ -\sqrt{2}(cs - \epsilon c^2) & s^2 & c^2 + 2cs\epsilon \end{pmatrix}. \quad (27)$$

Notice that, unlike in the two previous cases, the zeroth order texture (which obtains from (27) in the limit $\epsilon = 0$) depends on the mixing angle θ_{12} . This is because the exact degeneracy limit now corresponds to $m_2 = -m_1$ and not to $m_2 = m_1$, so that the mixing between ν_1 and ν_2 remains meaningful. Therefore the zeroth order texture in this case depends on the choice of the solution of the solar neutrino problem. For the SMA solution one has to choose $c = 1$, $s = 0$ or $c = 0$, $s = 1$; the corresponding zeroth order textures are $diag(-1, 1, 1)$ and the matrix with $m_{11} = m_{23} = m_{32}$, the rest of the elements being zero. Small nonvanishing θ_{12} results when the zeroth order textures are perturbed. Two further examples are

$$\begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \\ \sqrt{2} & 1 & 1 \end{pmatrix}, \quad \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}, \quad (28)$$

the first of which corresponds to the bimaximal mixing ($\theta_{12} = \theta_{23} = 45^\circ$) and the second is an example of the texture with a large but not maximal θ_{12} : $s = 1/\sqrt{3}$ ($\sin^2 2\theta_{12} = 8/9$).

⁴The case $m_1 \simeq -m_2 \simeq m_3$ can be obtained from the present one by the change of notation $\nu_1 \leftrightarrow \nu_2$.

We shall now consider one of the examples, namely, $c \simeq 0$, $s \simeq 1$, in more detail. The neutrino mass matrix for this case can be written as

$$m_L = m_0 \begin{pmatrix} 1 & \varepsilon & \varepsilon' \\ \varepsilon & \delta & 1 + \delta' \\ \varepsilon' & 1 + \delta' & \rho \end{pmatrix} \quad (29)$$

with small ε , ε' , δ , δ' and ρ . From eqs. (7)-(9) one finds

$$m_{1,2} \simeq \frac{m_0}{2} \left\{ \left(\frac{\rho + \delta}{2} - \delta' \right) \mp \sqrt{\left(2 + \delta' - \frac{\rho + \delta}{2} \right)^2 + 2(\varepsilon - \varepsilon')^2} \right\}, m_3 \simeq m_0 \left(1 + \delta' + \frac{\rho + \delta}{2} \right) \quad (30)$$

$$\tan 2\theta_{12} = -\frac{\varepsilon - \varepsilon'}{\sqrt{2}[1 + \delta'/2 - (\rho + \delta)/4]}, \quad \sin \theta_{13} = \frac{\varepsilon + \varepsilon'}{\sqrt{2}[(\rho + \delta)/2 + \delta']}. \quad (31)$$

The mixing angle θ_{13} depends on a ratio of the small parameters. It is not automatically small and care should be taken in order to satisfy the constraint (2). The mixing angle θ_{12} is generically small because the zeroth order texture corresponds to zero mixing. Thus, in this case only the SMA solution of the solar neutrino problem is possible. The neutrino mass splittings are

$$\tilde{\delta} \simeq m_0 [(\rho + \delta)/2 - \delta'], \quad \Delta \simeq m_0 [\rho + \delta]. \quad (32)$$

The condition that the lower-mass state out of the eigenstates ν_1 and ν_2 have the larger ν_e component is

$$\delta' > (\rho + \delta)/2. \quad (33)$$

The point at which the inequality is replaced by the equality is unstable with respect to small variations of the parameters. As follows from (32), the requirement of the correct hierarchy of the mass differences, $|\Delta| \gg |\tilde{\delta}|$, is equivalent to $\rho + \delta \simeq 2\delta'$. This means that in the present case the parameters are necessarily close to the instability point.

The value of the mixing angle θ_{13} as well as those of the neutrino mass splittings depend sensitively on the value of δ' . In order for $\tilde{\delta}/m_0$ and Δ/m_0 to be small so must be δ' , which we assume. Notice, however, that δ' is the difference of the two large matrix elements m_{23} and m_{11} . This means that a very small relative change of these elements can result in a drastic change of the whole pattern of neutrino masses and mixing. This situation is typical for all cases of quasi-degenerate neutrinos.

Finally, it should be noticed that if neutrinos are quasi-degenerate, care should be taken in order not to violate the limit (3) coming from the experiment on the neutrinoless double beta decay. It essentially constrains the m_{11} entry of the mass matrices. From the discussion in this subsection it follows that the common mass scale m_0 must satisfy this constraint in all quasi-degenerate cases except in the one of bimaximal mixing, for which the zeroth order texture is given by the first matrix in eq. (28). This case leads to the VO solution of the

solar neutrino problem. As follows from the second example in (28), the bound on m_0 can also be somewhat relaxed in the case of the LMA solution of the solar neutrino problem. In all quasi-degenerate cases in which the common mass scale has to satisfy the limit (3) neutrinos cannot constitute any appreciable part of the dark matter of the universe.

4 Conclusion

We have studied systematically the phenomenologically allowed structures of the neutrino mass matrix in the case of three light neutrino species. For both hierarchical and quasi-degenerate cases we considered neutrino mass matrices which can be obtained by perturbing the well known zeroth order textures.

Assuming maximal ν_μ - ν_τ mixing we derived simple analytic formulas which give neutrino masses and the remaining mixing angles in terms of the entries of the mass matrices. We have checked our approximate analytic expressions by performing the numerical diagonalization of neutrino mass matrix and found a very good agreement in each case.

We analyzed the stability of the neutrino masses and mixing angles with respect to small variations of the entries of the mass matrices, paying special attention to the small entries. In particular, we have studied the stability of the condition of having a larger ν_e component in the lower-mass state out of the eigenstates ν_1 and ν_2 responsible for the solar neutrino oscillations. This condition is important for the SMA and LMA solutions of the solar neutrino problem.

We have found that while the normal hierarchy case $|m_1|, |m_2| \ll |m_3|$ is in general very stable, the inverted hierarchy case $|m_3| \ll |m_1| \approx |m_2|$ is much less stable (except in the case of bimaximal mixing discussed in case (3) of sec. 3.2). The quasi-degenerate case suffers from a number of instabilities and therefore is the least natural one.

The author is grateful to G. C. Branco and M. N. Rebelo for useful discussions. This work was supported by Fundação para a Ciência e a Tecnologia through the grant PRAXIS XXI/BCC/16414/98 and also in part by the TMR network grant ERBFMRX-CT960090 of the European Union.

References

- [1] Super-Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett. 81 (1998) 1562; *ibid.* 82 (1999) 2644; *ibid.* 82 (1999) 5194; hep-ex/9908049.
- [2] Super-Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett. 81 (1998) 1158-1162; Erratum-*ibid.* 81 (1998) 4279; *ibid.* 82 (1999) 1810-1814; *ibid.* 82 (1999) 2430.

- [3] T. Kajita (SuperKamiokande Collaboration), Talk at *Beyond the Desert'99*, Ringberg Castle, Germany, June 6-12, 1999.
- [4] Kamiokande collaboration, K.S. Hirata *et al.*, Phys. Lett. B280 (1992) 146; Y. Fukuda *et al.*, Phys. Lett. B 335 (1994) 237; IMB collaboration, R. Becker-Szendy *et al.*, Nucl. Phys. Proc. Suppl. **38B**, 331 (1995); Soudan-2 collaboration, W.W.M. Allison *et al.*, Phys. Lett. B391 (1997) 491; M.C. Goodman, report ANL-HEP-CP-99-63; P. Bernardini (MACRO Collaboration), hep-ex/9906019.
- [5] Homestake Collaboration, B. T. Cleveland *et al.*, Astrophys. J. 496 (1998) 505; SAGE Collaboration, J. N. Abdurashitov *et al.*, astro-ph/9907113; Gallex Collaboration, P. Anselmann *et al.*, Phys. Lett. B447 (1999) 127; Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett. 77 (1996) 1683.
- [6] M. Fukugita, M. Tanimoto, T. Yanagida, Phys. Rev. D59:113016, 1999; Y. Nomura, T. Yanagida, Phys. Rev. D59:017303, 1999; C. H. Albright, S. M. Barr, Phys. Lett. B452 (1999) 287-293; hep-ph/9906297; B. Brahmachari, R. N. Mohapatra, Phys. Rev. D58:015003, 1998; G. Altarelli, F. Feruglio, JHEP 9811:021,1998; Phys. Lett. B451 (1999) 388; Z. Berezhiani, A. Rossi, JHEP 9903:002,1999; K. Hagiwara, N. Okamura, Nucl. Phys. B548 (1999) 60; K. S. Babu, B. Dutta, R.N. Mohapatra, Phys. Lett. B458 (1999) 93; M. Tanimoto, Phys. Lett. B456 (1999) 220; J. K. Elwood, N. Irges, P. Ramond, Phys. Rev. Lett. 81 (1998) 5064; S. Lola, G. G. Ross, Nucl. Phys. B553 (1999) 81; S. F. King, hep-ph/9904210; C. Wetterich, Phys. Lett. B451 (1999) 397; R. Barbieri, L. J. Hall, G. L. Kane, G. G. Ross, hep-ph/9901228; Y. Grossman, Y. Nir, Y. Shadmi, JHEP 9810:007, 1998; A. Ghosal, hep-ph/9905470; Yue-Liang Wu, hep-ph/9901320; M. Carena, J. Ellis, S. Lola, C.E.M. Wagner, hep-ph/9906362; W. Buchmuller, D. Delepine, F. Vissani, Phys. Lett. B459 (1999) 171; J. C. Romao, M. A. Diaz, M. Hirsch, W. Porod, J. W. F Valle, hep-ph/9907499; M. A. Diaz, J. Ferrandis, J.C. Romao, J. W. F. Valle, hep-ph/9906343; M. Tanimoto, T. Watari, T. Yanagida, hep-ph/9904338; W. Buchmuller, T. Yanagida, Phys. Lett. B445 (1999) 399-402; A. Yu. Smirnov, M. Tanimoto, Phys. Rev. D55 (1997) 1665; C. Jarlskog, M. Matsuda, S. Skadhauge, M. Tanimoto, Phys. Lett. B449 (1999) 240; A. S. Joshipura, S. D. Rindani, hep-ph/9907390; M. Fukugita, M. Tanimoto, T. Yanagida, Phys. Rev. D57 (1998) 4429; M. Jezabek, Y. Sumino, Phys. Lett. B440 (1998) 327; *ibid.*, B457 (1999) 139; E. Ma, D. P. Roy, Phys. Rev. D59:097702, 1999; E. Ma, D.P. Roy, U. Sarkar, Phys. Lett. B444 (1998) 391; F. Vissani, JHEP 9811:025, 1998; G. Altarelli, F. Feruglio, I. Masina, hep-ph/9907532 ; S. Davidson, S. F. King, Phys. Lett. B445 (1998) 191; J. Ellis, G. K. Leontaris, S. Lola, D. V. Nanopoulos, Eur. Phys. J. C9 (1999) 389; K. R. Dienes, E. Dudas, T. Gherghetta, hep-ph/9811428; G. Dvali, A. Yu. Smirnov, hep-ph/9904211; R. N. Mohapatra, S. Nandi, A. Perez-Lorenzana, hep-ph/9907520; F. Vissani, hep-ph/9708483; H. Georgi, S. L. Glashow, hep-ph/9808293; G. C. Branco, M. N. Rebelo, J. I. Silva-Marcos, Phys. Rev. Lett. 82 (1999) 683; hep-ph/9906368.

- [7] R. Barbieri, L. J. Hall, D. Smith, A. Strumia, N. Weiner, JHEP 9812:017, 1998; R. Barbieri, G. G. Ross, A. Strumia, hep-ph/9906470.
- [8] V. Barger, S. Pakvasa, T. J. Weiler, K. Whisnant, Phys. Lett. B437 (1998) 107-116; A. J. Baltz, A. S. Goldhaber, M. Goldhaber, Phys. Rev. Lett. 81 (1998) 5730; R. N. Mohapatra, S. Nussinov, Phys. Lett. B441 (1998) 299; Phys. Rev. D60:013002, 1999.
- [9] J. Ellis, S. Lola, Phys. Lett. B458 (1999) 310; J. A. Casas, J. R. Espinosa, A. Ibarra, I. Navarro, hep-ph/9904395; hep-ph/9905381; hep-ph/9906281; N. Haba N. Okamura, hep-ph/9906481; E. Ma, hep-ph/9907400; N. Haba, Y. Matsui, N. Okamura, M. Sug-iura, hep-ph/9908429.
- [10] For recent reviews see A. Yu. Smirnov, hep-ph/9901208; G. Altarelli, F. Feruglio, hep-ph/9905536; J. W. F. Valle, hep-ph/9907222; J. Ellis, hep-ph/9907458.
- [11] LSND Collaboration, C. Athanassopoulos *et al.*, Phys. Rev. Lett. 81 (1998) 1774; W. C. Louis (LSND Collaboration), Prog. Part. Nucl. Phys. 40 (1998) 151;
- [12] J. N. Bahcall, P. I. Krastev, A. Yu. Smirnov, Phys. Rev. **D 58** 096016 (1998); J. N. Bahcall, P. I. Krastev, A. Yu. Smirnov, hep-ph/9905220; M. C. Gonzalez-Garcia, P. C. de Holanda, C. Pena-Garay, J. W. F. Valle, hep-ph/9906469.
- [13] Particle Data Group, C. Caso *et al.*, Eur. Phys. J. C3 (1998) 1.
- [14] CHOOZ Collaboration, M. Apollonio *et al.*, hep-ex/9907037.
- [15] Heidelberg-Moscow experiment, L. Baudis *et al.*, hep-ex/9902014.